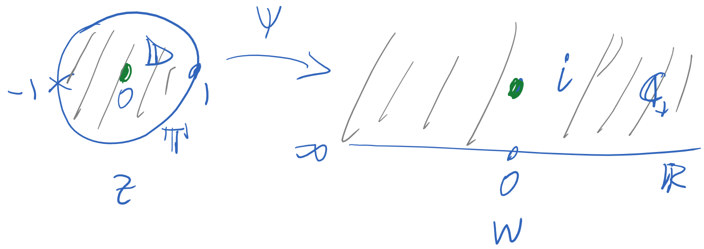
Recall:



$$\Psi(z) = i\,\frac{1-z}{1+z}$$

In the ball: $\mathbb{B} \xrightarrow{\ \Psi\ } \mathcal{U}$ Cayley map

$$z = (\underset{n}{z'}, \underset{1}{z_{n+1}}) \longmapsto w = (w', w_{n+1})$$

$$w_{n+1} = i\,\frac{1-z_{n+1}}{1+z_{n+1}}, \quad w' = \frac{2i\,z'}{1+z_{n+1}}$$

Obs. $z \overset{\Psi}{\longmapsto} w$ is projective: $\boxed{a \cdot z = \alpha} \Longleftrightarrow \boxed{b \cdot w = \beta}$

- $z_{n+1} = \dfrac{i - w_{n+1}}{i + w_{n+1}}, \quad z' = \dfrac{w'}{i + w_{n+1}}$

What's $\mathcal{U} := \Psi(\mathbb{B})$?

$$1 \underset{(=)}{>} |z_{n+1}|^2 + |z'|^2 = \frac{|i - w_{n+1}|^2 + |w'|^2}{|i + w_{n+1}|^2}$$

iff $|i + w_{n+1}|^2 \underset{(=)}{>} |i - w_{n+1}|^2 + |w'|^2$

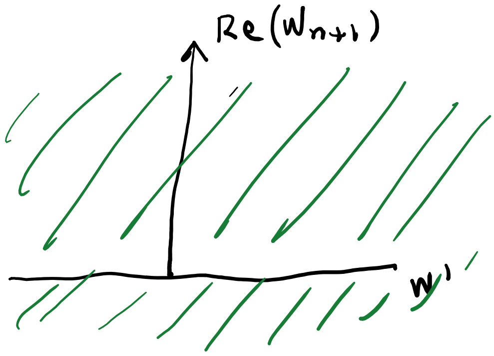
$$\|$$

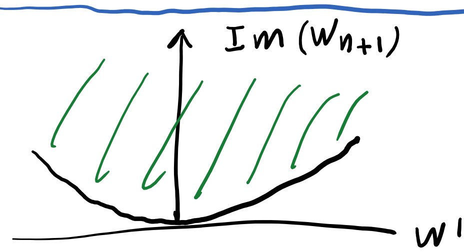
$$1 + |w_{n+1}|^2 + 2\,\mathrm{Re}(i\,w_{n+1}) \qquad\qquad 1 + |w_{n+1}|^2 - 2\,\mathrm{Re}(i\,w_{n+1}) + |w'|^2$$

$$4\,\mathrm{Im}(w_{n+1}) \underset{(=)}{>} |w'|^2$$

- $\mathcal{U} = \left\{ w = (w', w_{n+1}) : \dfrac{|w'|^2}{4} < \mathrm{Im}(w_{n+1}) \right\}$

SIEGEL DOMAIN





Change of names: $w = (w', w_{n+1}) \longleftrightarrow P = (z, z_{n+1})$

$\boxed{\mathcal{U}:\ \mathrm{Im}(z_{n+1}) \underset{(=)}{>} \dfrac{|z|^2}{4}}$ $\quad z \in \mathbb{C}^n$

Automorphisms of $\mathcal{U}$ $\mathcal{U} \xrightarrow{\ \ } \mathcal{U}$ biholomorphic

Rotations: $(z, z_{n+1}) \longmapsto (Az, z_{n+1})$; $A \in SU(n)$

Dilations: $(z, z_{n+1}) \longmapsto (\lambda z, \lambda^2 z_{n+1})$: $\lambda > 0$

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Translations: I start with $z \mapsto z + a$, $a \in \mathbb{C}^n$ and I fix $(n+1)^{th}$ coordinate accordingly:

defining function

$$\frac{|z+a|^2}{4} - \left(\frac{|z|^2}{4} - \mathrm{Im}(z_{n+1})\right) = \frac{|a|^2}{4} + \frac{\mathrm{Re}(\bar{a} z)}{2} + \mathrm{Im}(z_{n+1})$$

$$= \mathrm{Im}\left(i\frac{|a|^2}{4} + i\frac{\bar{a} z}{2} + z_{n+1} \right)$$

$$\tau_{[a,0]}(z, z_{n+1}) = \left(z + a,\ i\frac{|a|^2}{4} + i\frac{\bar{a} z}{2} + z_{n+1} \right)$$

More generally

$$\tau_{[a,s]}(z, z_{n+1}) = \left(z + a,\ i\frac{|a|^2}{4} + i\frac{\bar{a} z}{2} + s + z_{n+1} \right)$$

$[a,s] \in \mathbb{C}^n \times \mathbb{R}$

There is an inversion as well (see notes).

Heisenberg coordinates in U

$$U \ni (z, z_{n+1}) = [\underbrace{z}_{\mathbb{C}^n}, \underbrace{\mathrm{Re}(z_{n+1})}_{\mathbb{R}};\ \underbrace{\overbrace{\mathrm{Im}(z_{n+1}) - \frac{|z|^2}{4}}^{h}}_{[0,+\infty)}] = [z, t; h]$$

$U = \mathbb{C}^n \times \mathbb{R} \times [0, +\infty)$

$$[z, t; h] = \left(\underbrace{z}_{\mathbb{C}^n},\ \underbrace{t + i\left(h + \frac{|z|^2}{4}\right)}_{\mathbb{C}}\right)$$

Obs. By construction, $\tau_{[a,s]}([z, t; h]) = [z_1, t_1; h]$ preserves h

Proposition. In Heisenberg coordinates,

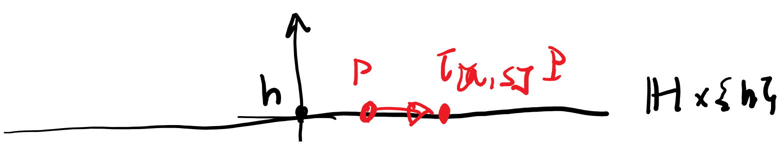
$$\tau_{[a,s]}[z, t; h] = \left[a + z,\ s + t - \frac{1}{2}\mathrm{Im}(\bar{a} z);\ h\right]$$

Pf: $\tau_{[a,s]}(z, z_{n+1}) = \left(z + a,\ s + z_{n+1} + i\frac{|a|^2}{4} + i\frac{\bar{a} z}{2} \right)$

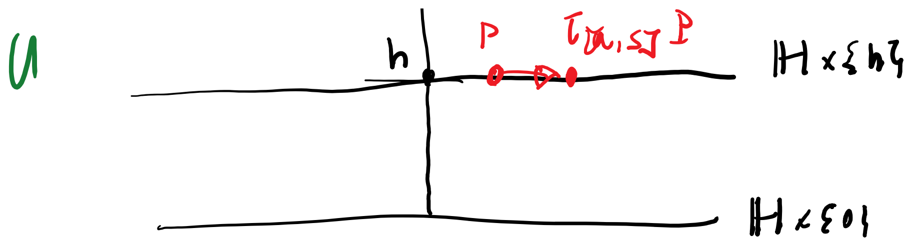
$$= \left[z + a,\ \mathrm{Re}\left(s + z_{n+1} + i\frac{\bar{a} z}{2} \right);\ h \right]$$

$$= \left[z + a,\ s + t - \frac{1}{2}\mathrm{Im}(\bar{a} z);\ h \right] \quad \blacksquare$$

U



'horocycle'



"horocycli"

$\mathbb{H} \ni [z,t]$

$\cong$

$\mathbb{C}\times\mathbb{R}$

$\mathbb{H}$ is a Lie group: noncommutative

$$[z,t]\cdot[w,s] := [z+w,\ t+s-\tfrac{1}{2}\mathrm{Im}(\bar{z}w)]$$

$$[z,t]\cdot[0,0] = [z,t] = [0,0]\,[z,t]$$

$$[z,t]^{-1} = [-z,-t]$$

It acts on each "leaf" $\mathbb{H}\times\{h\}$.

$$\mathbb{H}\times U \longrightarrow U$$

$$(g,P) = ([z,t],[w,s;h]) \longmapsto [[z,t]\cdot[w,s];h] = g\cdot P$$

Action is simple and transitive on leaves

$$[z,t;h] = [z,t]\cdot[0,0;h]$$

"Action" means: $g_1\cdot(g_2\cdot P) = (g_1\cdot g_2)\cdot P$

We have a Lie group $\mathbb{H}$ (acting on U from the left)

- Lie algebra?
- Invariant (Haar) measure?

$E\subseteq\mathbb{H}$: $\quad m(gE) \underset{\substack{\uparrow\\ \text{left invariant}}}{=} m(E) \underset{\substack{\uparrow\\ \text{right invariant}}}{=} m(Eg)$

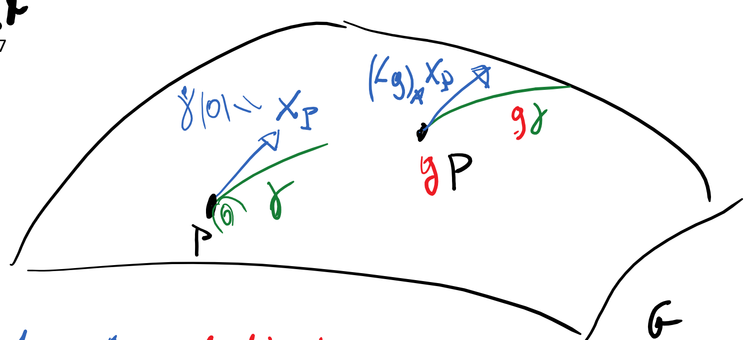
Lebesgue m on $\mathbb{C}^n\times\mathbb{R}$ is biinvariant

7 7

- Lie algebra

$$G \xrightarrow{L_g} G$$

$$P \longmapsto g\cdot P$$



X: vector field, is left invariant if

$$\boxed{(L_g)_* X_P = X_{gP}}$$

$$e_j = \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} j\text{-th} \qquad z = x+iy$$

$$[z,t]\,[\tau e_j, \underset{= x(1)}{0}] = [z+\tau e_j,\ t-\tfrac{1}{2}\mathrm{Im}(\bar{z}\tau e_j)]$$

$$[z,t][\tau e_j, \underset{=\gamma(\tau)}{0}] = \left[z+\tau e_j,\ t - \tfrac{1}{2}\operatorname{Im}(\bar z \tau e_j)\right] \quad (j) \qquad z = x+iy$$

$$= \left[z+\tau e_j,\ t+\tfrac{1}{2}\tau y_j\right] \xrightarrow{\frac{d}{d\tau}\big|_{\tau=0}} \left[e_j, \tfrac{y_j}{2}\right] = \partial_{x_j} + \tfrac{y_j}{2}\partial_t$$

$$[z,t][\tau i e_j, 0] \longrightarrow \left[i e_j, -\tfrac{x_j}{2}\right] = \partial_{y_j} - \tfrac{x_j}{2}\partial_t$$

$$[z,t][0,\tau] \longrightarrow \partial_t$$

$j=1\dots n$

horizontal
$$\begin{cases} X_j = \partial_{x_j} + \frac{y_j}{2}\partial_t \\ Y_j = \partial_{y_j} - \frac{x_j}{2}\partial_t \end{cases}$$
vertical
$$\left\{ T = \partial_t \right.$$

$\longrightarrow$ $j=1\dots n$

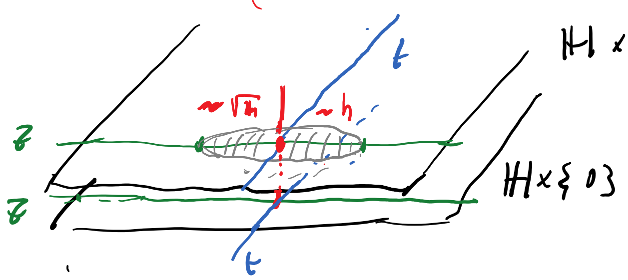
$[X_j, Y_j] = -T$
and all other Lie brackets vanish.

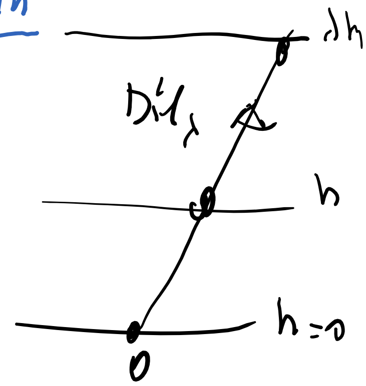
Obs. The fields X_j, Y_j are killed by
$$dt - \omega = dt - \frac{y\,dx - x\,dy}{2}$$

$$(dt-\omega)X_l = \left[dt - \tfrac{1}{2}\textstyle\sum (y_j\,dx_j - x_j\,dy_j)\right]\left(\partial_{x_l} + \tfrac{y_l}{2}\partial_t\right)$$
$$= \frac{y_l}{2}\,dt(\partial_t) - \frac{1}{2}\frac{y_l}{2}\,dx_l(\partial_{x_l}) = 0 \quad \blacksquare$$

Bergman metric in U using Heisenberg coordinates.

Theorem. $$ds^2 = \frac{|dz|^2}{\underset{(\sqrt{h})^2 =}{h}} + \frac{(dt-\omega)^2 + dh^2}{h^2}$$





How do we prove it?

- Verify that under $\mathbb{B} \xrightarrow{\Psi} U$, $w \longmapsto [z,t;h]$

$$ds^2\big|_{[0,0;1]} = |dz|^2$$

• Move it around using translations ($\mathbb{H}$) and dilations.

Orthonormal system?

$$\tilde{X}_j = \sqrt{h}\left(\partial_{x_j} + \frac{y_j}{2}\partial_t\right) = \sqrt{h}\, X_j$$

$$\tilde{Y}_j = \sqrt{h}\left(\partial_{y_j} - \frac{x_j}{2}\partial_t\right) = \sqrt{h}\, Y_j$$

$$\tilde{T} = h\,\partial_t = h\,T$$

$$H = h\,\partial_h = h\,H$$

Length of a curve

$$\dot{\gamma}(\tau) = \sum_{j=1}^{n}\Big(\underbrace{a_j(\tau)\sqrt{h}}_{\tilde{a}_j}\overbrace{X_j}^{\tilde{X}_j} + \underbrace{b_j(\tau)\sqrt{h}}_{\tilde{b}_j} Y_j\Big) + \underbrace{c(\tau)h}_{\tilde{c}}T + \underbrace{d(\tau)h}_{\tilde{d}}H$$

$\tilde{c}(\tau)$

$$\Rightarrow \text{Length}(\gamma) = \int_I \sqrt{\sum_{j=1}^{n}\left(a_j^2(\tau) + b_j^2(\tau)\right) + c(\tau)^2 + d(\tau)^2}\, d\tau$$

$$= \int_I \sqrt{\sum_{j=1}^{n}\frac{\tilde{a}_j^2 + \tilde{b}_j^2}{h} + \frac{\tilde{c}^2}{h^2} + \frac{\tilde{d}^2}{h^2}}\, d\tau$$

Suppose γ lives on $\mathbb{H}\times\{h\}$: $d = 0$

$$\sqrt{h}\cdot\text{Length}(\gamma) = \int_I \sqrt{\sum_{j=1}^{n}\tilde{a}_j^2 + \tilde{b}_j^2 + \frac{\tilde{c}^2}{h}}\, dt$$

$h \to 0$

huge penalization!

$\text{Subriem-Length}(\gamma)$

$= +\infty$ unless $\tilde{c} = 0$

$$\dot{\gamma} = \tilde{a}(\tau)X + \tilde{b}Y + 0\cdot T + \cancel{0 H}$$